

ASYMPTOTICS OF GENERALISED TRINOMIAL COEFFICIENTS

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ABSTRACT. It is shown how to obtain an asymptotic expansion of the generalised central trinomial coefficient $[x^n](x^2 + bx + c)^n$ by means of singularity analysis, thus proving a conjecture of Zhi-Wei Sun.

In [6], Zhi-Wei Sun proposes a number of conjectural formulas for multiples of $1/\pi$ involving the generalised central trinomial coefficient $T_n(b, c)$, which is defined by

$$T_n(b, c) = [x^n](x^2 + bx + c)^n.$$

Besides the conjectural series, for example

$$\sum_{k=0}^{\infty} \frac{30k+7}{(-256)^k} \binom{2k}{k}^2 T_k(1, 16) = \frac{24}{\pi},$$

he also presents conjectures regarding the asymptotic behaviour of $T_n(b, c)$ as $n \rightarrow \infty$, namely:

Conjecture 1 (Sun [6, Conjecture 5.1]). *For $b > 0$ and $c > 0$, we have*

$$T_n(b, c) = \frac{(b + 2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}} \left(1 + \frac{b - 4\sqrt{c}}{16n\sqrt{c}} + O\left(\frac{1}{n^2}\right) \right)$$

as $n \rightarrow \infty$. If $c > 0$ and $b = 4\sqrt{c}$, then

$$\frac{T_n(b, c)}{\sqrt{c}^n} = \frac{3 \cdot 6^n}{\sqrt{6n\pi}} \left(1 + \frac{1}{8n^2} + \frac{15}{64n^3} + \frac{21}{32n^4} + O\left(\frac{1}{n^5}\right) \right).$$

Finally, if $c < 0$ and $b \in \mathbb{R}$, then

$$\lim_{n \rightarrow \infty} \sqrt[n]{|T_n(b, c)|} = \sqrt{b^2 - 4c}.$$

The aim of this little note is to show how the conjecture can be proven by a standard application of singularity analysis [2].

The special cases $d = b^2 - 4c = 0$ **and** $b = 0$. If the discriminant is 0, then $T_n(b, c)$ essentially reduces to a central binomial coefficient:

$$T_n(b, c) = [x^n](x^2 + bx + b^2/4)^n = [x^n](x + b/2)^{2n} = (b/2)^n \binom{2n}{n}.$$

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In this case one can obtain an asymptotic expansion by means of Stirling's formula:

$$\begin{aligned} T_n(b, b^2/4) &= (b/2)^n \cdot \frac{(2n/e)^{2n} \cdot \sqrt{4\pi n}}{(n/e)^{2n} \cdot 2\pi n} \cdot \left(1 + \frac{1}{24n} + \frac{1}{1152n^2} - \frac{139}{414720n^3} + O(n^{-4})\right) \\ &\quad \cdot \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O(n^{-4})\right)^{-2} \\ &= \frac{(2b)^n}{\sqrt{\pi n}} \left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} + O(n^{-4})\right). \end{aligned}$$

Similarly, if $b = 0$, we obtain

$$T_n(0, c) = \begin{cases} c^{n/2} \binom{n}{n/2} & n \text{ even,} \\ 0 & n \text{ odd,} \end{cases}$$

and we can use Stirling's formula again.

The case $c > 0$. Let us now assume that $b > 0$ (since $T_n(b, c) = (-1)^n T_n(-b, c)$, we can focus on this case), $c > 0$ and $d = b^2 - 4c \neq 0$. Then we can write

$$T_n(b, c) = d^{n/2} L_n(b/\sqrt{d}),$$

where L_n is the n -th Legendre polynomial. It is thus sufficient to study the asymptotic behaviour of the Legendre polynomials. The Laplace-Heine formula states that

$$L_n(x) \sim \frac{(x + \sqrt{x^2 - 1})^{n+1/2}}{\sqrt{2n\pi} \sqrt{x^2 - 1}}$$

as $n \rightarrow \infty$ if $-1 < x < 1$, which already yields the main term in Conjecture 1 if $c > 0$. For our purposes, we mostly need the generating function

$$\sum_{n=0}^{\infty} L_n(x) t^n = \frac{1}{\sqrt{1 - 2xt + t^2}},$$

from which we get

$$\begin{aligned} F(t) &= \sum_{n=0}^{\infty} T_n(b, c) t^n = \sum_{n=0}^{\infty} L_n(b/\sqrt{d}) (\sqrt{d} t)^n \\ &= \frac{1}{\sqrt{1 - 2bt + dt^2}} = \frac{1}{\sqrt{1 - 2bt + (b^2 - 4c)t^2}}. \end{aligned}$$

The formula even remains valid when $d \leq 0$. This function has two singularities at the zeros of the polynomial $1 - 2bt + (b^2 - 4c)t^2$. If $b > 0$ and $c > 0$, then these singularities are at $t_1 = 1/(b + 2\sqrt{c})$ and at $t_2 = 1/(b - 2\sqrt{c})$, and t_1 is closer to the origin. We now invoke singularity analysis (see [3, Chapter VI] for a detailed explanation of this technique) to obtain the asymptotic behaviour of $T_n(b, c)$ from the expansion around the dominant singularity t_1 :

$$F(t) = \frac{1}{2c^{1/4}} (t_1 - t)^{-1/2} - \frac{b^2 - 4c}{16c^{3/4}} (t_1 - t)^{1/2} + \frac{3(b^2 - 4c)^2}{256c^{5/4}} (t_1 - t)^{3/2} + \dots$$

We can translate each term according to the rule

$$\begin{aligned} C(1 - t/t_1)^{-\alpha} &\mapsto C \binom{n + \alpha - 1}{n} t_1^{-n} \\ &= \frac{C t_1^{-n} n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \frac{a(a-1)}{2n} + \frac{a(a-1)(a-2)(3a-1)}{24n^2} + O(n^{-3}) \right) \end{aligned}$$

to obtain

$$T_n(b, c) = \frac{t_1^{-n-1/2}}{2c^{1/4}\sqrt{n\pi}} \left(1 + \frac{b - 4\sqrt{c}}{16\sqrt{cn}} + \frac{(3b - 4\sqrt{c})^2}{512cn^2} + O(n^{-3}) \right).$$

This proves the first part of Conjecture 1, even with an additional term in the asymptotic expansion. By including further terms in the expansion around t_1 and in the expansion of the binomial coefficients $\binom{n+\alpha-1}{n}$, one can obtain even more precise asymptotic formulas. Let us illustrate this in the case that $b = 4\sqrt{c}$, when we get

$$F(t) = \frac{1}{\sqrt{1 - 2bt + 3b^2t/4}} = \frac{1}{\sqrt{(1 - bt/2)(1 - 3bt/2)}}.$$

The expansion around the dominant singularity $t_1 = 2/(3b)$ is, with $u = 1 - 3bt/2$,

$$F(t) = \sqrt{\frac{3}{2}} \left(u^{-1/2} - \frac{1}{4}u^{1/2} + \frac{3}{32}u^{3/2} - \frac{5}{128}u^{5/2} + \frac{35}{2048}u^{7/2} - \frac{63}{8192}u^{9/2} + \dots \right).$$

Moreover, a more precise asymptotic expansion of the Gamma function yields

$$\begin{aligned} C \binom{n + \alpha - 1}{n} t_1^{-n} &= \frac{C t_1^{-n} n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \frac{\alpha(\alpha-1)}{2n} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} \right. \\ &\quad + \frac{\alpha^2(\alpha-1)^2(\alpha-2)(\alpha-3)}{48n^3} + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)(15\alpha^3 - 30\alpha^2 + 5\alpha + 2)}{5760n^4} \\ &\quad \left. + \frac{\alpha^2(\alpha-1)^2(\alpha-2)(\alpha-3)(\alpha-4)(\alpha-5)(3\alpha^2 - 7\alpha - 2)}{11520n^5} + O(n^{-6}) \right). \end{aligned}$$

Hence the terms in the expansion of $F(t)$ translate as follows:

$$\begin{aligned} u^{-1/2} &\mapsto \frac{t_1^{-n} n^{-1/2}}{\Gamma(1/2)} \left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{1024n^3} - \frac{21}{32768n^4} - \frac{399}{262144n^5} + O(n^{-6}) \right), \\ u^{1/2} &\mapsto \frac{t_1^{-n} n^{-3/2}}{\Gamma(-1/2)} \left(1 + \frac{3}{8n} + \frac{25}{128n^2} + \frac{105}{1024n^3} + \frac{1659}{32768n^4} + O(n^{-5}) \right), \\ u^{3/2} &\mapsto \frac{t_1^{-n} n^{-5/2}}{\Gamma(-3/2)} \left(1 + \frac{15}{8n} + \frac{385}{128n^2} + \frac{4725}{1024n^3} + O(n^{-4}) \right), \\ u^{5/2} &\mapsto \frac{t_1^{-n} n^{-7/2}}{\Gamma(-5/2)} \left(1 + \frac{35}{8n} + \frac{1785}{128n^2} + O(n^{-3}) \right), \\ u^{7/2} &\mapsto \frac{t_1^{-n} n^{-9/2}}{\Gamma(-7/2)} \left(1 + \frac{63}{8n} + O(n^{-2}) \right), \\ u^{9/2} &\mapsto \frac{t_1^{-n} n^{-11/2}}{\Gamma(-9/2)} (1 + O(n^{-1})). \end{aligned}$$

Putting everything together, we arrive at

$$T_n(b, c) = \frac{3 \cdot 6^n}{\sqrt{6n\pi}} \left(1 + \frac{1}{8n^2} + \frac{15}{64n^3} + \frac{21}{32n^4} + \frac{315}{128n^5} + O\left(\frac{1}{n^6}\right) \right),$$

which is the second part of Conjecture 1, even with one extra term.

The case $c < 0$. If $b > 0$ and $c < 0$, then the generating function $F(t)$ has two dominant singularities, since $t_1 = \frac{1}{b+2i\sqrt{-c}}$ and $t_2 = \frac{1}{b-2i\sqrt{-c}}$ have the same distance from the origin. The singularity at t_1 is of the form

$$F(t) \sim \sqrt{\frac{b+2i\sqrt{-c}}{4i\sqrt{-c}}} (1-t/t_1)^{-1/2},$$

and at t_2 , it is

$$F(t) \sim \sqrt{\frac{b-2i\sqrt{-c}}{-4i\sqrt{-c}}} (1-t/t_2)^{-1/2}.$$

Combining the contributions of the two, we obtain

$$\begin{aligned} T_n(b, c) &\sim \frac{1}{\Gamma(1/2)\sqrt{n}} \cdot \left(\sqrt{\frac{b+2i\sqrt{-c}}{4i\sqrt{-c}}} \cdot t_1^{-n} + \sqrt{\frac{b-2i\sqrt{-c}}{-4i\sqrt{-c}}} \cdot t_2^{-n} \right) \\ &= \frac{1}{2(-c)^{1/4}\sqrt{\pi n}} \cdot \left(\frac{1-i}{\sqrt{2}} (b+2i\sqrt{-c})^{n+1/2} + \frac{1+i}{\sqrt{2}} (b-2i\sqrt{-c})^{n+1/2} \right) \\ &= \frac{1}{(-c)^{1/4}\sqrt{\pi n}} (b^2 - 4c)^{n/2+1/4} \cos((n+1/2)\phi - \pi/4), \end{aligned}$$

where ϕ is given by $e^{i\phi} = (b+2i\sqrt{-c})/\sqrt{b^2-4c}$. The third part of Conjecture 1 follows immediately. Again, one could also get a further asymptotic expansion by taking more terms in the expansions around t_1 and t_2 into account.

Some final remarks. An alternative approach to these results would be the saddle point method (see Chapter VIII of [3], cf. in particular Example VIII.11). Instead of the central trinomial coefficients, one can also treat other trinomial coefficients, see for instance [4, 5] for an application to random walks. Higher polynomial coefficients have of course been studied as well, see for example [1, p.77].

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